

Charged perfect fluids in the presence of a cosmological constant

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Abstract

We consider the static and spherically symmetric field equations of general relativity for charged perfect fluid spheres in the presence of a cosmological constant. Following work by Florides (1983) we find new exact solutions of the field equations, and discuss their mass radius ratios. These solutions, for instance, require the charged Nariai metric to be the vacuum part of the spacetime. We also find charged generalizations of the Einstein static universe and speculate that the smallness problem of the cosmological constant might become less problematic if charge is taken into account.

1 Introduction

Ever since Schwarzschild constructed the first exact interior solution of the Einstein field equations, static and spherically symmetric matter solutions have been of great interest. Additional degrees of freedom of the matter were included at later stages, like the cosmological constant, anisotropic pressure and charge. An entire new class of interior Schwarzschild like solutions was obtained by Florides [11], these particular solutions were distinct since they featured a vanishing radial stress T_r^r . This Schwarzschild like solution was later generalized to include a non-zero charge distribution, see [12, 14, 16]. In each case the new interior solution was matched to an exterior Reissner-Nordström solution, or the exterior Schwarzschild solution in the absence of charge.

Static charged fluid spheres had been studied in detail by Efinger (1965), for instance, who was interested in understanding charged particles in general relativity, see also [2, 3, 8, 10, 14, 15, 19]. Note that the spacetimes obtained

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so far were not singularity free. Then Mehra (1979) managed to construct an interior solution and avoid a singularity. Florides (1983) also considered charged perfect fluid solutions, these solutions described the complete field, interior and exterior. The neutral Florides interior solution was then extended further by [20] to include the effects of a non zero cosmological constant Λ . This solution was matched to the exterior Schwarzschild-de Sitter (Kottler) solution. Since then not much attention has been devoted to analyze charged perfect solutions in the presence of a cosmological constant.

In this paper we follow [11] and impose the restriction that the fluid component of the matter is isotropic. In this way we obtain new classes of interior charged solutions in the presence of a cosmological constant. A particular class of these solutions, namely the charged analogue of the interior Nariai solution is discussed in greater detail. We show how to match this solution to the exterior charged Nariai solution. Note that the exterior is not asymptotically flat and thus has a somewhat special place in the family of Reissner-Nordström de Sitter solutions. We are also able to construct charged generalizations of the Einstein static universe where the pressure is monotonically decreasing or increasing between the two regular centers.

2 Field equations

We consider the system of Einstein-Maxwell equations with cosmological term

$$G_{ij} + \Lambda g_{ij} = 8\pi T_{ij}, \quad T_{ij} = M_{ij} + E_{ij},$$

$$\partial_j(\sqrt{-g}F^{ij}) = \sqrt{-g}J^i, \quad \partial_{[i}F_{jk]} = 0,$$

where we set $G = c = 1$. The matter energy-momentum tensor is denoted by M_{ij} , and the electromagnetic energy-momentum tensor E_{ij} is given by

$$E_{ij} = \frac{1}{4\pi} \left(F_i^k F_{jk} - \frac{1}{4} g_{ij} F^{mn} F_{mn} \right).$$

The 4-current density vector J^i is defined by

$$J^i = 4\pi\sigma u^i.$$

The proper charge density is denoted by σ and u^i is the 4-velocity satisfying $u^i u_i = -1$.

Following the notation of [13], we consider the static and spherically metric in the form

$$ds^2 = -e^{a(r)} dt^2 + e^{b(r)} dr^2 + r^2 d\Omega^2, \quad (2.1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element of the unit sphere.

Using this metric ansatz and the above defined 4-current density vector, the only non-vanishing component of the electromagnetic tensor F_{ij} becomes

$$F_{rt} = \frac{e^{(a+b)/2}}{r^2} q, \quad q = 4\pi \int_0^r \sigma e^{b/2} r'^2 dr'. \quad (2.2)$$

Therefore, the resulting components of E_i^j are

$$E_t^t = E_r^r = -E_\theta^\theta = -E_\phi^\phi = -\frac{q^2}{8\pi r^4}. \quad (2.3)$$

As for the matter we choose an anisotropic perfect fluid

$$M_i^j = \text{diag}(-\rho, p, p_\perp, p_\perp). \quad (2.4)$$

The resulting (tt) and (rr) components of the Einstein-Maxwell field equations and the energy-momentum conservation equation take the following form

$$\frac{1}{r^2} \frac{d}{dr} (r - r e^{-b}) - \Lambda = 8\pi\rho + \frac{q^2}{r^4}, \quad (2.5)$$

$$\frac{e^{-b}}{r^2} + \frac{a' e^{-b}}{r} - \frac{1}{r^2} + \Lambda = 8\pi p - \frac{q^2}{r^4}, \quad (2.6)$$

$$p' + \frac{a'}{2}(\rho + p) + \frac{2}{r}(p - p_\perp) - \frac{qq'}{4\pi r^4} = 0. \quad (2.7)$$

Therefore, this system consists of three independent equations for six unknown functions $\{a, b, \rho, p, p_\perp, q\}$. Therefore the system is under-determined and additional physical or mathematical assumptions need to be taken into account in order to close the set of equations.

For example, in the neutral case $q \equiv 0$, with an isotropic perfect fluid $p = p_\perp$ there are three equations and four unknowns. In that case the most physical approach is to consider a barotropic equation of state for the matter, $\rho = \rho(p)$ which closes the set of equations. Another popular approach motivated by astrophysical observations is the prescription of a density profile $\rho(r)$. However, this usually leads to a singular center and is therefore problematic from a rigorous point of view.

In the charged case, one can impose various physical conditions to decrease the number of unknown functions. One can for instance assume that the total energy momentum tensor is isotropic

$$T_r^r = T_\theta^\theta = T_\phi^\phi, \quad (2.8)$$

which yields the condition

$$p - p_\perp = \frac{1}{4\pi} \frac{q^2}{r^4}. \quad (2.9)$$

Therefore, this assumption leads to a relation between the anisotropy (difference between radial and tangential pressure) and the electromagnetic field. Note that in the absence of charges this immediately leads to an isotropic perfect fluid.

The second natural assumption is to consider an isotropic perfect fluid

$$M_r^r = M_\theta^\theta = M_\phi^\phi, \quad (2.10)$$

which simply gives

$$p = p_{\perp} . \quad (2.11)$$

Note that more conditions need to be imposed to close the system of equations.

A useful linear combination is the sum of equations (2.5) and (2.6) which yields

$$a' + b' = 8\pi r e^b (\rho + p) . \quad (2.12)$$

The first field equation (2.5) can be formally integrated and we obtain

$$\begin{aligned} e^{-b} &= 1 - \frac{1}{r} \int_0^r \left(8\pi\rho + \frac{q^2}{r'^4} \right) r'^2 dr' - \frac{\Lambda}{3} r^2 \\ &= 1 - \frac{2m_g}{r} + \frac{q^2}{r^2} - \frac{\Lambda}{3} r^2 , \end{aligned} \quad (2.13)$$

where we used (2.2) and where

$$m_g = m_i + m_q = \int_0^r 4\pi\rho r'^2 dr' + \int_0^r 4\pi r' \sigma q e^{b/2} dr' . \quad (2.14)$$

The field equations can be combined into the useful Tolman-Oppenheimer-Volkoff equation, which is a differential equation for the pressure expressed in terms of the other matter quantities. Eliminating a' from (2.7) using (2.6) and (2.13), we find

$$p' = -r \frac{(4\pi p + \frac{m_g}{r^3} - \frac{q^2}{r^4} - \frac{\Lambda}{3})(\rho + p)}{1 - \frac{2m_g}{r} + \frac{q^2}{r^2} - \frac{\Lambda}{3} r^2} - \frac{2}{r}(p - p_{\perp}) - \frac{qq'}{4\pi r^4} , \quad (2.15)$$

which reduces to the well-known Tolman-Oppenheimer-Volkoff equation when $q \equiv 0$, $\Lambda = 0$ and $p \equiv p_{\perp}$.

If the interior solution is of finite extent, we denote total quantities with capital letters, this means M, Q denote the total mass and total charge of the solution, respectively. We use r_b to denote the boundary of the object, defined as the vanishing pressure surface. Therefore, if the exterior spacetime is the Reissner-Nordström de Sitter spacetime, its metric is given by

$$\begin{aligned} ds^2 &= - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2 \right) dt^2 \\ &\quad + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2 \right)^{-1} dr^2 + r^2 d\Omega^2 , \end{aligned} \quad (2.16)$$

where $r \geq r_b$.

The main aim of this work is to find new exact solutions of the field equations. Therefore, in view of Eq. (2.13) one particular choice would be to consider the case when

$$8\pi\rho + \frac{q^2}{r^4} = 8\pi\mu \equiv \text{constant} . \quad (2.17)$$

This condition has been first considered in [9] where a charged dust was considered. It has further been exploited by [13]. In this case the interior metric takes the simpler form

$$e^{-b} = 1 - \frac{r^2}{R^2}, \quad (2.18)$$

$$\frac{1}{R^2} = \frac{8\pi\mu}{3} + \frac{\Lambda}{3}. \quad (2.19)$$

Combining equations (2.19) and (2.18) with (2.17) yields the metric function e^{-b} in the form

$$e^{-b} = 1 - \frac{8\pi\rho}{3}r^2 - \frac{q^2}{3r^2} - \frac{\Lambda}{3}r^2. \quad (2.20)$$

3 Solutions of the field equations

3.1 Florides' solution

Let us consider an isotropic total energy-momentum tensor (Condition 1) which gives (2.9). Inserting this into the conservation equation (2.7) gives

$$p' + \frac{1}{2}a'(\rho + p) + \frac{1}{2\pi} \frac{q^2}{r^5} - \frac{1}{4\pi} \frac{qq'}{r^4} = 0. \quad (3.1)$$

Now we substitute a' using (2.12) and use the fact that

$$-\frac{d}{dr} \left(\frac{q^2}{8\pi r^4} \right) = \frac{1}{2\pi} \frac{q^2}{r^5} - \frac{1}{4\pi} \frac{qq'}{r^4},$$

we arrive at the equation

$$p' - \frac{1}{2}b'(\rho + p) + 4\pi r e^b (\rho + p)^2 - \frac{d}{dr} \left(\frac{q^2}{8\pi r^4} \right) = 0.$$

Finally, we introduce the quantity $z = \rho + p$ which results in

$$z' - \frac{1}{2}b'z + 4\pi r e^b z^2 - \frac{d}{dr} \left(\rho + \frac{q^2}{8\pi r^4} \right) = 0. \quad (3.2)$$

If we assume the term in the bracket to be a constant (Condition 2), see the condition (2.17), then this equation simplifies considerably which we will assume henceforth, which yields

$$z' - \frac{1}{2}b'z + 4\pi r e^b z^2 = 0. \quad (3.3)$$

Next, making use of (2.18) we have

$$b' = \frac{2r/R^2}{1 - r^2/R^2}, \quad (3.4)$$

and therefore

$$z' - \frac{r/R^2}{1 - r^2/R^2}z + \frac{4\pi r}{1 - r^2/R^2}z^2 = 0, \quad (3.5)$$

which can be integrated using separation of variables

$$z = \frac{1}{4\pi R^2} \left[1 + C \sqrt{1 - \frac{r^2}{R^2}} \right]^{-1}, \quad (3.6)$$

where C is a constant of integration.

The other metric function a follows from (2.12)

$$e^{a/2} = \frac{1 + C \sqrt{1 - \frac{r^2}{R^2}}}{1 + C}, \quad (3.7)$$

where we fixed the other constant of integration by requiring $e^a(r = 0) = 1$. Therefore we found the complete metric. Note that in a static spacetime we can always rescale the time coordinate arbitrarily.

3.2 Towards new solutions

In [13], the constant of integration C was determined by the condition that the pressure vanishes at some radius $p(r_b) = 0$. Since this condition requires the existence of a vanishing pressure hypersurface, we follow a different approach and fix the constant of integration using the central values of pressure and energy density.

Let us define the central pressure and energy density to be $p_c = p(r = 0)$ and $\rho_c = \rho(r = 0)$, respectively. Then

$$\rho_c + p_c = z_c(r = 0) = \frac{1}{4\pi R^2(1 + C)}, \quad (3.8)$$

$$C = -1 + \frac{1}{4\pi R^2(\rho_c + p_c)}. \quad (3.9)$$

In the chosen coordinate system there is a coordinate singularity when $r \rightarrow R$. The spatial part of the metric is a 3-sphere, hence it is natural to introduce the third Euler angle of the sphere by

$$r = R \sin \chi, \quad 1 - \frac{r^2}{R^2} = \cos^2 \chi. \quad (3.10)$$

In these new coordinates the interior metric takes the form

$$ds^2 = - \left(\frac{1 + C \cos \chi}{1 + C} \right)^2 dt^2 + R^2 d\chi^2 + R^2 \sin^2 \chi d\Omega^2, \quad (3.11)$$

which covers the entire 3-sphere. Note that the matter part of the solution does not necessarily have to ‘fill’ this three sphere. In fact, solutions analogous

to the Schwarzschild interior solution have their vanishing pressure surface located before the equator of this three sphere and thus this new coordinate is not important in understanding them. However, in the presence of a cosmological constant one can construct solutions where the matter part occupies more than half the three sphere. Understanding these solutions requires a proper coordinate system covering the entire space.

Let us now insert the constant of integration expressed as central values into the interior metric

$$ds^2 = -\left(4\pi R^2(\rho_c + p_c)(1 - \cos \chi) + \cos \chi\right)^2 dt^2 + R^2 d\chi^2 + R^2 \sin^2 \chi d\Omega^2. \quad (3.12)$$

While we have been able to find the metric, in order to find all unknown functions, we have to impose one additional condition to close the system. As this final condition (Condition 3) we assume $q = er^2 = eR^2 \sin^2 \chi$, where e has dimension charge/length². This condition immediately implies that the effective energy density is constant, by virtue of (2.17). Note that the resulting energy momentum tensor is regular everywhere, however, the proper charge density σ blows up like $1/r$ near the center by virtue of (2.2).

This allows us to find the radial pressure of the matter configuration as a function of the radius or the third Euler angle. For instance, using $p = z - \rho$ we obtain

$$p = \frac{(\rho_c + p_c)(4\pi\rho_c - \Lambda - e^2) - \rho_c(4\pi(\rho_c + 3p_c) - \Lambda - e^2)\cos \chi}{(4\pi(\rho_c + 3p_c) - \Lambda - e^2)\cos \chi - 12\pi(\rho_c + p_c)}. \quad (3.13)$$

This pressure function vanishes at $\chi = \chi_b$ where

$$\cos \chi_b = \frac{(\rho_c + p_c)(4\pi\rho_c - \Lambda - e^2)}{\rho_c(4\pi(\rho_c + 3p_c) - \Lambda - e^2)}. \quad (3.14)$$

Note that χ_b does not have to exist. In fact, we will construct solutions where the pressure never vanishes.

Since we have found an expression of the form $\chi_b = \chi_b(\rho_c, p_c, \Lambda, e^2)$, we can also express the central pressure in terms of the other quantities. For this we find

$$p_c = \frac{\rho_c(1 - \cos \chi_b)(4\pi\rho_c - \Lambda - e^2)}{4\pi\rho_c(3\cos \chi_b - 1) + \Lambda + e^2}. \quad (3.15)$$

Since we are only interested in matter distributions with regular center, we assume the central pressure to be finite which is equivalent to the positivity of the denominator of (3.15). Therefore, we find

$$\cos \chi_b > \frac{1}{3} - \frac{\Lambda}{12\pi\rho_c} - \frac{e^2}{12\pi\rho_c}, \quad (3.16)$$

which is a generalization of the well-known Buchdahl inequality. To see this, we go back to the radial coordinate r and use that $\cos(\arcsin x) = \sqrt{1-x^2}$. This gives

$$\cos\left(\arcsin \frac{r_b}{R}\right) = \sqrt{1 - \frac{r_b^2}{R^2}} = e^{-b(r_b)/2}, \quad (3.17)$$

by virtue of (2.18). Our Condition 3 states that $q/r^2 = e$, inserting this into (2.20) leads to

$$e^{-b(r_b)} = 1 - \frac{8\pi\rho}{3}r_b^2 - \frac{e^2}{3}r_b^2 - \frac{\Lambda}{3}r_b^2. \quad (3.18)$$

Let us finally denote $M = 4\pi/3 \rho r_b^3$, then equation (3.16) becomes

$$\sqrt{1 - \frac{2M}{r_b} - \frac{e^2}{3}r_b^2 - \frac{\Lambda}{3}r_b^2} > \frac{1}{3} - \frac{\Lambda}{12\pi\rho_c} - \frac{e^2}{12\pi\rho_c}, \quad (3.19)$$

which is the familiar form of the Buchdahl inequality, see also [1, 7]. Letting $\Lambda, e^2 \rightarrow 0$ gives the textbook result

$$\sqrt{1 - \frac{2M}{r_b}} > \frac{1}{3}, \quad \frac{2M}{r_b} < \frac{8}{9}. \quad (3.20)$$

3.3 Classes of new solutions

Based on the pressure function (3.13) we can straightforwardly identify new classes of solutions. Firstly, let us assume that the pressure vanishes at the equator of the three sphere, this means when $\chi_b = \pi/2$. This type of solution occurs when the relation $\Lambda + e^2 = 4\pi\rho_c$ holds. When $\Lambda + e^2 > 4\pi\rho_c$, the pressure vanishes after the equator of the three sphere. If we furthermore want the pressure to vanish before the second center of the three sphere, we must impose $4\pi\rho_c < \Lambda + e^2 < 8\pi\rho_c(\rho_c + 2p_c)/(2\rho_c + p_c)$.

Next, we can encounter situation where the pressure does not vanish anywhere and the solution has a second regular center, this means we assume the pressure at the second center to be finite. In that case $8\pi\rho_c(\rho_c + 2p_c)/(2\rho_c + p_c) < \Lambda + e^2 < 8\pi(2\rho_c + 3p_c)$. If Λ exceeds this upper limit the pressure diverges. In this class of solutions, there is one special solution where the pressure is constant. This happens when $\Lambda + e^2 = 4\pi(\rho_c + 3p_c)$ which is a charged generalization of the Einstein static universe. In the following Section, we will discuss these new types of solutions in detail.

4 New exact solutions

4.1 Solutions with exterior Nariai metric

Let us assume $\Lambda + e^2 = 4\pi\rho_c$. In this case the pressure function (3.13) vanishes when $\chi_b = \pi/2$, the equator of the three sphere. Now we need to identify the

correct vacuum solution matched at the vanishing pressure surface. It is clear that the vacuum solution in this case cannot be part of the Reissner-Nordström de Sitter spacetime. The only other static and spherically symmetric spacetime with charge and cosmological constant is the charged Nariai metric, we collect some basic facts of this solution in the Appendix, A.

This metric is given by

$$ds^2 = \frac{1}{A} \left(-(\alpha \sin \psi + \beta \cos \psi)^2 dt^2 + d\psi^2 \right) + \frac{1}{B} d\Omega^2, \quad (4.1)$$

where the cosmological constant and the total charge are related to A and B by

$$\Lambda = \frac{1}{2}(A + B), \quad Q^2 = \frac{1}{2}(B - A), \quad (4.2)$$

$$A = \Lambda - Q^2, \quad B = \Lambda + Q^2. \quad (4.3)$$

This is the most general form of the charged Nariai solution, the constants α and β can be chosen arbitrarily and will be fixed by our matching conditions. In the following we will show that metric (4.1) is the correct exterior for (3.11) or (3.12) when the matching is performed at the $\chi = \pi/2$ hypersurface.

There are two equivalent approaches to matching two metrics at a given hypersurface. One can either match the first and second fundamental forms or one can introduce Gauss coordinates relative to the hypersurface and then show that metric is continuous and differentiable at the matching surface. We will follow the latter approach since both metrics are almost in the correct form.

Let us introduce a new coordinate $\zeta = R\chi$ for (3.11), which we will call the interior metric

$$ds_{\text{int}}^2 = - \left(\frac{1 + C \cos(\zeta/R)}{1 + C} \right)^2 dt^2 + d\zeta^2 + R^2 \sin^2(\zeta/R) d\Omega^2. \quad (4.4)$$

Let us also introduce a new coordinate $\zeta = \psi/\sqrt{A}$ for metric (4.1), the exterior metric

$$ds_{\text{ext}}^2 = - \frac{1}{A} (\alpha \sin(\sqrt{A}\zeta) + \beta \cos(\sqrt{A}\zeta))^2 dt^2 + d\zeta^2 + \frac{1}{B} d\Omega^2. \quad (4.5)$$

The matching surface (vanishing pressure surface) is located at $\zeta = \pi R/2$.

Continuity of the metric yields

$$R^2 = \frac{1}{B} \quad (4.6)$$

$$\frac{1}{1+C} = \frac{1}{\sqrt{A}} (\alpha \sin(\sqrt{A}\pi R/2) + \beta \cos(\sqrt{A}\pi R/2)) \quad (4.7)$$

while continuity of the first derivative implies

$$- \frac{2}{1+C} \frac{C}{R} = \frac{2}{A} (\alpha \sin(\sqrt{A}\pi R/2) + \beta \cos(\sqrt{A}\pi R/2)) \times (\alpha \sqrt{A} \cos(\sqrt{A}\pi R/2) - \beta \sqrt{A} \sin(\sqrt{A}\pi R/2)). \quad (4.8)$$

It is straightforward to solve these equations for α and β . The most elegant approach seems to be to firstly use that $R = 1/\sqrt{B}$, and secondly use the second equation for the first factor of the right-hand side of the third. This gives

$$\sin\left(\frac{\sqrt{A}}{\sqrt{B}}\frac{\pi}{2}\right)\alpha + \cos\left(\frac{\sqrt{A}}{\sqrt{B}}\frac{\pi}{2}\right)\beta = \frac{\sqrt{A}}{1+C}, \quad (4.9)$$

$$\cos\left(\frac{\sqrt{A}}{\sqrt{B}}\frac{\pi}{2}\right)\alpha - \sin\left(\frac{\sqrt{A}}{\sqrt{B}}\frac{\pi}{2}\right)\beta = -\frac{2C}{R}. \quad (4.10)$$

In the absence of charge, $A = B = \Lambda$ and these equations simplify considerably since the trigonometric functions become either 1 or 0. Let us denote $\xi = \sqrt{A/B}\pi/2$, then one can write the latter equations as the following simple linear system of equations

$$\begin{pmatrix} \sin \xi & \cos \xi \\ \cos \xi & -\sin \xi \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sqrt{A}/(1+C) \\ -2C/R \end{pmatrix}. \quad (4.11)$$

The matrix on the left is its own inverse and thus α and β are given by

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sin \xi & \cos \xi \\ \cos \xi & -\sin \xi \end{pmatrix} \begin{pmatrix} \sqrt{A}/(1+C) \\ -2C/R \end{pmatrix}. \quad (4.12)$$

Therefore, we have shown that the interior and the exterior metric can be matched, the metric being C^1 at the surface. Without further assumptions, this cannot be improved, as can be seen by the following argument. The energy density inside the charged star is constant, while it is zero in the vacuum region. Therefore, the energy-momentum tensor has a jump. By virtue of the Einstein-Maxwell field equations, the Einstein tensor must have a jump too. Since it contains the second derivatives of the metric, the metric is at most C^1 .

We would also like to note that these interior solutions can be interpreted as matter solutions which require the Bertotti-Robinson spacetime [4, 18] to be the electro-vacuum part of the manifold if we assume a vanishing cosmological constant.

4.2 Solutions with black hole event horizons

If $4\pi\rho_c < \Lambda + e^2 < 8\pi\rho_c(\rho_c + 2p_c)/(2\rho_c + p_c)$, then the pressure vanishes after the equator of the three sphere but before its second center. In this case the vanishing pressure surface is in a region where the area group orbits are decreasing. The vacuum part of this spacetime is part of the Reissner-Nordström de Sitter solution. However, this region contains the black hole event horizon. This unusual class of solutions has been studied in the past in the absence of charge, see [5, 6]. Since these solutions appear to have little physical relevance, we will not discuss them further.

4.3 Generalized Einstein static universes

We now assume that $\Lambda + e^2$ is in the range $8\pi\rho_c(\rho_c + 2p_c)/(2\rho_c + p_c) < \Lambda + e^2 < 8\pi(2\rho_c + 3p_c)$. In this case the pressure is strictly positive. The spacetime has two regular centers and it is natural to refer to such spacetimes as generalizations of the Einstein static universes.

Within these solutions, there is the charged generalization of the original Einstein static universe. If we choose $\Lambda + e^2 = 4\pi(\rho_c + 3p_c)$, the prefactors of the trigonometric functions in the pressure (3.13) vanish and the pressure is constant. The radius of this charged Einstein static universe is $1/R^2 = 4\pi(\rho_c + p_c)$.

It should be noted that in all of the above solutions, the cosmological constant and the charge parameter appear in the combination $\Lambda + e^2$ and one can regard this as an effective cosmological constant, note that $e^2 \geq 0$. For the charged Einstein static universe in particular this means that we can construct such a matter configuration without cosmological constant. In this case the charge acts as an external force that pulls the matter apart and balances the gravitational attraction.

Solutions with two centers are symmetric with respect to $\chi = \pi/2$. To see this, let us denote the pressures at the two centers by p_1 and p_2 , respectively, then

$$p_1 := p(\chi = 0) = p_c, \quad (4.13)$$

$$p_2 := p(\chi = \pi) = \frac{2\rho_c(4\pi\rho_c - \Lambda - e^2) + p_1(16\pi\rho_c - \Lambda - e^2)}{\Lambda + e^2 - 8\pi(2\rho_c + 3p_1)}. \quad (4.14)$$

The latter equation can be solved for $\Lambda + e^2$, which then becomes a function of ρ_c, p_1, p_2

$$\Lambda + e^2 = \frac{8\pi\rho_c(\rho_c + 2p_1 + 2p_2) + 3p_1p_2}{2\rho_c + p_1 + p_2}. \quad (4.15)$$

This allows us to eliminate $\Lambda + e^2$ from the pressure function (3.13) and we arrive at

$$p(\chi)[\rho_c, p_1, p_2] = \frac{\rho_c(p_1 + p_2) + 2p_1p_2 + \rho_c(p_1 - p_2)\cos\chi}{2\rho_c + p_1 + p_2 - (p_1 - p_2)\cos\chi}. \quad (4.16)$$

The pressure vanishes at χ_b where

$$\cos\chi_b = \frac{(\rho_0 - p_1)(p_2(2p_1 - 1) - p_1)}{p_1(\rho_0 + p_1) + p_2(2\rho_0p_1 - \rho_0 + p_1)}. \quad (4.17)$$

The aforementioned symmetry property of the pressure function can now easily be expressed mathematically, namely we have

$$p(\pi/2 - \gamma)[\rho_c, p_1, p_2] = p(\pi/2 + \gamma)[\rho_c, p_2, p_1]. \quad (4.18)$$

5 Conclusions

The aim of this paper was to construct new classes of exact solutions of Einstein's field equations in a static and spherically symmetric setting. We considered a charged perfect fluid to be the matter source. One class of new solutions is an interior solution which requires the charged Nariai metric to describe the vacuum part of this spacetime. The other class generalizes the Einstein static universe. These solutions are characterized by two regular centers and a non-uniform pressure function (4.16). They possess an additional symmetry property, namely, the pressure is point-symmetric with respect to the middle between the two centers.

In our particular setting the cosmological constant Λ and the charge e always appear in the combination $\Lambda + e^2$ which we can either view as an effective charge or an effective cosmological constant. It is the latter point of view which might be of interest in the context of cosmology. If these results are not an artefact of our setting, then a small overall charge density throughout the universe might be viewed as a natural explanation of a small cosmological constant.

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A The charged Nariai solution

The charged Nariai solution forms a two dimensional subspace of the Reissner-Nordström de Sitter class of solutions. The inner, outer and cosmological horizons are all equivalent, sometimes called degenerate. Note that this implies

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3} = 0, \quad (\text{A.1})$$

the breakdown of the the (r, t) coordinate system which describes only part of the complete spacetime. We will follow the approach in [17] where it is noted that the Nariai spacetime can be viewed as a 4 dimensional submanifold of a flat 6 dimensional Lorentzian manifold with product structure. Let us consider

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2, \quad (\text{A.2})$$

such that

$$-x_0^2 + x_1^2 + x_2^2 = \frac{1}{A}, \quad x_3^2 + x_4^2 + x_5^2 = \frac{1}{B}, \quad (\text{A.3})$$

with $A \neq B = \text{const.}$ and $A + B = 2\Lambda$. Thus this spacetime is a direct product of two 3 dimensional manifolds, one being a 3 sphere and the other one being a hyperbolic 3 space. To make this relation explicit, let us consider the following parametrization

$$\begin{aligned} x_0 &= \frac{1}{\sqrt{A}} \frac{1}{\sqrt{\alpha^2 + \beta^2}} (\alpha \sin \psi + \beta \cos \chi) \sinh(\sqrt{\alpha^2 + \beta^2} t), \\ x_1 &= \frac{1}{\sqrt{A}} \frac{1}{\sqrt{\alpha^2 + \beta^2}} (\alpha \sin \psi + \beta \cos \chi) \cosh(\sqrt{\alpha^2 + \beta^2} t), \\ x_2 &= \frac{1}{\sqrt{A}} \frac{1}{\sqrt{\alpha^2 + \beta^2}} (\beta \sin \psi - \alpha \cos \chi), \\ x_3 &= \frac{1}{\sqrt{B}} \sin \theta \cos \phi, \\ x_4 &= \frac{1}{\sqrt{B}} \sin \theta \sin \phi, \\ x_5 &= \frac{1}{\sqrt{B}} \cos \theta. \end{aligned} \quad (\text{A.4})$$

Metric (A.2) then gives metric (4.1).